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The extension of the Miles–Howard theorem to compressible fluids

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A statically stable, gravitationally stratified compressible fluid containing a parallel shear flow is examined for stability against infinitesimal adiabatic perturbations. It is found that the Miles–Howard theorem of incompressible fluids may be generalized to this system, so that $n^2 \geqslant \frac{1}{4}U'^2$ throughout the flow is a sufficient condition for stability. Here n^2 is the Brunt–Väissälä frequency and U' is the vertical gradient of the flow speed. Howard's upper bound on the growth rate of an unstable mode also generalizes to this compressible system.

1. Introduction

The conditions for the stability of a gravitationally stratified incompressible inviscid fluid containing a parallel shear flow were obtained by Miles (1961) and Howard (1961), thus verifying the general validity of results which emerged from earlier investigations of specific flow fields. It is then of interest to know whether the Miles–Howard theory carries over in a simple form to compressible fluids.

Warren (1968) obtained a sufficient condition for stability of the compressible flow, but his expression is weaker than the expected generalization of the theory, and as he indicated, is not the optimum condition.

In fact we show in this paper that the stability of a compressible fluid against adiabatic perturbations may be treated in a manner entirely analogous to that used by Howard, with correspondingly simple results. Actually this procedure only establishes the existence or non-existence of complex eigenvalues, and is thus not the complete investigation of the stability problem.

2. Equations of motion of the system

Let (x, y, z) form a right-handed set of co-ordinates with $\mathbf{g}$ the gravitational acceleration acting in the negative z direction. The unperturbed fluid is defined by a mass density $\rho_0(z)$, a pressure $p_0(z)$ and a parallel shear flow $\mathbf{U}_0(z)$ with $\mathbf{U}_0(z).\,\hat{\mathbf{z}} = 0$. In the usual manner we define the sound speed C and the Brunt–Väissälä frequency n.

$$C^2(z) = \left(\frac{\partial p}{\partial \rho}\right)_s, \tag{2.1}$$

$$n^2(z) = -g\left(\frac{\rho_0'}{\rho_0} + \frac{g}{C^2}\right), \tag{2.2}$$

where a prime represents differentiation with respect to z.

Ignoring heat conduction and viscous effects the equations of motion governing development of the system are:

the Euler equation

$$\rho \frac{D\mathbf{U}}{Dt} = \rho \mathbf{g} - \nabla p; \tag{2.3}$$

the equation of continuity

$$\frac{D\rho}{Dt} + \rho \nabla . \mathbf{U} = 0; \tag{2.4}$$

and the equation of adiabatic pressure changes

$$\frac{Dp}{Dt} = \left(\frac{\partial p}{\partial \rho}\right)_s \frac{D\rho}{Dt} = C^2 \frac{D\rho}{Dt}, \tag{2.5}$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{U} . \nabla \quad . \tag{2.6}$$

We express p, ρ, and $\mathbf{U}$ as sums of their unperturbed values and (infinitesimal) perturbations, and then make the usual linearization in the perturbation amplitudes. Assuming the form

$$A(x, y, z, t) = A(z) \exp (i[\omega t - \mathbf{k} . (\mathbf{x} + \mathbf{y})]) \tag{2.7}$$

with k real and $\omega = \omega_r + i\omega_i \quad (\omega_i < 0)$

for each scalar component A of the perturbation fields produces a set of simultaneous differential equations which may be reduced to a single second-order linear homogeneous equation for one of the amplitudes A. Accordingly we chose to obtain an equation in $w(z)$, the z component of the fluid velocity.

$$\left[\frac{r\Omega^2 q'}{k^2 - (\Omega/C)^2}\right]' + r(n^2 - \Omega^2) q = 0, \tag{2.8}$$

where

$$q = \frac{w(z)}{i\Omega} \exp\left(-g \int^z C^{-2}(z')\, dz'\right), \tag{2.9}$$

$$r = \rho_0(z) \exp\left(2g \int^z C^{-2}(z')\, dz'\right), \tag{2.10}$$

$$\Omega = (\omega - \mathbf{k} . \mathbf{U}_0(z)) \tag{2.11}$$

(cf. Warren 1968 and Howard 1961).

For a statically stable fluid, n^2 and r are both real and positive. Only the component of $\mathbf{U}_0(z)$ parallel to $\mathbf{k}$ enters (2.8), and for convenience we denote this component by $U(z)$.

We accept two forms of boundary conditions on w. Either the domain z terminates at a rigid parallel wall on which q must be zero, or outside some finite region of z the unperturbed fluid assumes constant values for C^2 and U. In the latter case the outgoing radiation conditions are applied, and it is found that $r^{\frac{1}{2}}q$ goes to zero exponentially as $|z| \to \infty$, a result which could no doubt be considerably generalized.

3. The stability conditions

As Ω has a non-zero imaginary part we may uniquely define a branch of $\Omega^{\frac{1}{2}}$ and make the substitution

$$q = \Omega^{-\frac{1}{2}}\phi \tag{3.1}$$

in (2.8) to obtain

$$\left[\frac{r\Omega\phi'}{k^2(1-(\Omega/kC)^2)}\right]' - \frac{(\Omega')^2 r}{4k^2\Omega(1-(\Omega/kC)^2)}\phi - \frac{1}{2}\left[\frac{r\Omega'}{k^2(1-(\Omega/kC)^2)}\right]'\phi + r\frac{n^2-\Omega^2}{\Omega}\phi = 0. \tag{3.2}$$

Multiplying (3.2) by ϕ^* and integrating over the domain of z gives, with minor rearrangement of terms,

$$\int dz\left\{\left[\frac{r\Omega\phi^*\phi'}{k^2(1-(\Omega/kC)^2)}\right]' - \frac{r\Omega|\phi'|^2}{k^2(1-(\Omega/kC)^2)} - \frac{(\Omega')^2 r|\phi|^2}{4k^2\Omega(1-(\Omega/kC)^2)} - \frac{1}{2}\left[\frac{r\Omega'|\phi|^2}{k^2(1-(\Omega/kC)^2)}\right]' + \frac{1}{2}\frac{r\Omega'}{k^2(1-(\Omega/kC)^2)}(\phi\phi^*)' + r\frac{n^2-\Omega^2}{\Omega}|\phi|^2\right\} = 0. \tag{3.3}$$

The first and fourth terms integrate directly and are seen from the boundary conditions to yield zero identically. Extracting the imaginary part of (3.3) from the remaining terms gives us the equation

$$\omega_i\int dz\frac{r}{|1-(\Omega/kC)^2|^2k^2}\left\{(1+(|\Omega|/kC)^2)\,|\phi'|^2 + (\Omega')^2\frac{1+(|\Omega|/kC)^2}{4k^2C^2}|\phi|^2 - \Omega'\frac{\Omega_r}{k^2C^2}(\phi\phi^*)' + |1-(\Omega/kC)^2|^2k^2|\phi|^2\right\} + \omega_i\int dz\, r\frac{n^2-(\Omega'/2k)^2}{|\Omega|^2}|\phi|^2 = 0, \tag{3.4}$$

where Ω_r is the real part of Ω. Considering the first integral we remark that

$$\begin{aligned}(1+(|\Omega|/kC)^2)\,|\phi'|^2 + (\Omega')^2\frac{1+(|\Omega|/kC)^2}{4k^2C^2}|\phi|^2 - \Omega'\frac{\Omega_r}{k^2C^2}(\phi\phi^*)' \\ = \left[(1+(|\Omega|/kC)^2)^{\frac{1}{2}}|\phi'| - |\Omega'|\frac{(1+(|\Omega|/kC)^2)^{\frac{1}{2}}}{2|k|C}|\phi|\right]^2 \\ + \left[\frac{1+(|\Omega|/kC)^2}{|\Omega|/|kC|}|\Omega'|\frac{|\Omega|\,|\phi'|}{k^2C^2}|\phi| - \Omega'\frac{\Omega_r(\phi\phi^*)'}{k^2C^2}\right].\end{aligned} \tag{3.5}$$

Examining the second square bracket, we observe that

$$\frac{1+(|\Omega|/kC)^2}{|\Omega|/|kC|}|\Omega| > 2|\Omega_r|, \tag{3.6}$$

while

$$(\phi\phi^*)' \leqslant 2|\phi|\,|\phi'|. \tag{3.7}$$

Hence the expression on the right-hand side of (3.5) is a non-negative real number p^2 (say), so that (3.4) may be rewritten as

$$\omega_i\int dz\,\frac{r}{|1-(\Omega/kC)^2|^2k^2}\{|1-(\Omega/kC)^2|^2k^2|\phi|^2+p^2\} + \omega_i\int dz\, r\frac{n^2-(\Omega'/2k)^2}{|\Omega|^2}|\phi|^2 = 0. \tag{3.8}$$

As stated earlier, r and n^2 are positive, so that if ω_i is non-zero it is necessary that somewhere within the domain of z, $(\Omega'/2k)^2$ is greater than n^2.

Thus a sufficient condition for stability of the system is that $U'^2 \leqslant 4n^2$ throughout the flow. This condition is the obvious generalization of the Miles–Howard theorem, and since

$$n^2 = -g(\rho_0'/\rho_0 + g/C^2), \tag{3.9}$$

reduces to their expression in the limit $C^2 \to \infty$.

4. The upper bound on the growth rate

For ω_i non-zero, (3.8) gives the inequality

$$\int dz\, r \frac{\frac{1}{4}(U')^2 - n^2}{|\Omega|^2} |\phi|^2 \geqslant \int dz\, r |\phi|^2 \tag{4.1}$$

or since

$$\frac{\frac{1}{4}U'^2 - n^2}{|\Omega|^2} \leqslant \frac{[\frac{1}{4}U'^2 - n^2]_{\max}}{\omega_i^2},$$

$$\frac{[\frac{1}{4}U'^2 - n^2]_{\max}}{\omega_i^2} \int dz\, r |\phi|^2 \geqslant \int dz\, r |\phi|^2 \tag{4.2}$$

and

$$\omega_i^2 \leqslant [\tfrac{1}{4}U'^2 - n^2]_{\max}, \tag{4.3}$$

which is the generalization of the bound obtained by Howard (1961).

5. The semi-circle theorem

If we return to (2.8), multiplying it by q^*, integrating over the domain z, extracting the real and imaginary parts, and generally following precisely the procedure used by Howard (1961, §3), we derive Eckart's semi-circle theorem (Eckart 1963).

We have not, however, found the generalization of Rayleigh's theorem, as following the method of Howard, §5, does not seem to lead to a useful result in this case.

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